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Time-Domain Interpolation Using the Parametric DFT

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The main directions of application of digital signal processing methods in classical discrete spectral analysis based on the discrete Fourier transform are considered. An algorithm for interpolating a time signal based on a discrete Fourier transform is given. The disadvantages of the existing interpolation algorithm for the time signal based on the discrete Fourier transform are revealed. Based on the analysis of the matrix structure of the inverse discrete Fourier transform, a modified parametric discrete Fourier transform is proposed. The properties of the basis of the modified parametric discrete Fourier transform are investigated.

Keywords: digtal Fourier transform, inversed digital Fourier transform, parametric fast Fourier transform, modified parametric discrete exponential functions.

INTRODUCTION

One of the main areas of application of digital signal processing (DSP) methods is the classical discrete spectral analysis (CDSA) based on the discrete Fourier transform (DFT) [1–31].

Algebraic form of DFT:

$$S_N(k) = \frac{1}{N}\sum_{n=0}^{N-1} x(n) W_N^{kn}, \ \ k = \overline{0, N-1}; \tag{1}$$

where $x(n)$ is a discrete signal, $n = \overline{0, N-1}$; $S_N(k)$ – DFT coefficients, the set of which determines the amplitude-frequency and phase-frequency signal spectra (the coefficients of the DFT are often called bins); $W_N = \exp\left(-j\frac{2\pi}{N}\right)$, $k = \overline{0, N-1}$.

TIME-DOMAIN INTERPOLATION USING THE DFT

In the papers [2] time-domain interpolation using the FFT is described. According to the papers, we compute the FFT of an *N*-point $x(t)$ time sequence to produce its $S(k)$ frequency-domain samples.

Next, we stuff $N(r-1)$ zeros in the middle of $S_N(k)$ to yield the Nr-length $S_{\text{int}}(k)$ frequency samples, where Nr is an integer power of two.

Then we perform an Nr-point inverse FFT on $S_{\text{int}}(k)$ to obtain the interpolated-by-*r* $x_{\text{int}}(n)$ times samples.

Using this frequency-domain zero stuffing to implement time-domain signal interpolation involves two important issues.

The first issue: to ensure the interpolated $x_{\text{int}}(n)$ time sequence is real only, conjugate symmetry must be maintained in the zero-stuffed $S_{\text{int}}(k)$ frequency samples.

Here's the second issue regarding time-domain real signal interpolation: this exact interpolation algorithm provides correct results only if the original $x(n)$ sequence is periodic within its full time interval. If $S(k)$ exhibits any spectral leakage, like the signals with which we usually work, the interpolated $x_{\text{int}}(n)$ sequence can contain noticeable amplitude errors in its beginning and ending samples. As shown in Fig. 1(a) [2] where an $N = 24$ sample $x(n)$ sequence is interpolated by $r = 2$. In that figure, squares (both white and black) represent the 48-point interpolated $x_{\text{int}}(n)$ sequence, white squares are the original $x(n)$ samples, and circles represent the exactly correct interpolated sample values. (In the center portion of the figure, the circles are difficult to see because they're hidden behind the squares.) The interpolation error, the difference between the correct samples and $x_{\text{int}}(n)$, is given in Fig. 1(*b*) [2].

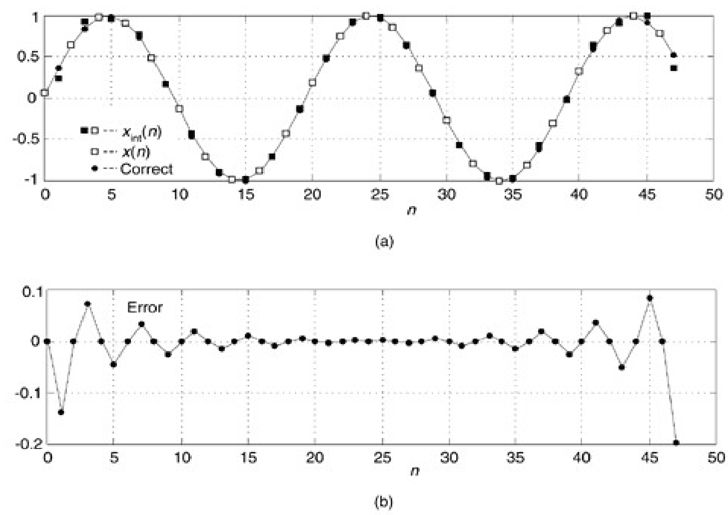

Figure 1. Interpolation results for $N = 24$, $r = 2$:
interpolated $x_{\text{int}}(n)$, original $x(n)$, and correct interpolation (*a*); interpolation error (*b*)

DFT and IDFT are the base of CDSA of signals in the matrix form, and described by the following relationships [1–5]:

$$\text{DFT} \Rightarrow S_N = \frac{1}{N} F_N X_N; \tag{2}$$

$$\text{IDFT} \Rightarrow X_N = \frac{1}{N} F_N{}^* S_N. \tag{3}$$

where $X_N = \left[x(0), x(1), \ldots, x(N-1)\right]^T$ – representation of a discrete signal $x(n)$, $n = \overline{0, N-1}$, in the form of a vector N-dimensional linear space; T is a sign of transposition; $S_N = \left[s(0), s(1), \ldots, s(N-1)\right]^T$ is the vector of expansion coefficients X_N in the system of discrete exponential functions (DEF): $def(p,l) = \exp\left(-j\frac{2\pi}{N}pl\right) = W_N^{pl}$; which is given by the matrix F_N:

$$F_N = \begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & . & . & (N-1) \end{array} \;\; n \\ \begin{array}{c} 0 \\ 1 \\ . \\ . \\ (N-1) \\ k \end{array} & \begin{bmatrix} 1 & 1 & . & . & 1 \\ 1 & W_N^{1} & . & . & W_N^{(N-1)} \\ . & . & . & . & . \\ . & . & . & . & . \\ 1 & W_N^{(N-1)} & . & . & W_N^{(N-1)(N-1)} \end{bmatrix} \end{array}. \tag{4}$$

Zero-padding technique in the frequency domain (method of adding the spectrum of a discrete signal $x(n)$ ($n = \overline{0, (N-1)}$) with $N \cdot (r-1)$ zero-valued samples ($r = \overline{1, 2\ldots}$) in the frequency domain (ZPFD) [1, 2]. With ZPFD, the discrete spectrum S_N of the real signal $x(n)$, $n = \overline{0, N-1}$, is represented in the form of a vector of M-dimensional ($M = N \cdot r$) linear space:

$$S_M = \left[s(0), \ldots, s(N/2-1), \underbrace{0, \ldots, 0}_{N(r-1)}, s(N/2), \ldots, s(N-1)\right]^T. \tag{5}$$

As a result of performing the IDFT of the vector X_M, the values of the discrete-frequency Fourier transform (DFFT) of the discrete signal $x(n)$ at M points uniformly distributed on the time axis are obtained.

The discrete-frequency Fourier transform is the inverse z-transform $S_N(k)$:

$$x(t) = \sum_{k=0}^{N-1} S_N(k) \cdot \exp(+j2\pi \cdot k \cdot t). \tag{6}$$

The zero-padding technique, applied to the spectrum of a discrete signal $x(n)$ with N samples in the frequency domain (ZPFD) allows us to find the values of the MDPF-P only for the parameter values $\xi = 0, 1/r, \ldots, (r-1)/r$. ZPFD has the following significant shortcomings, which manifest themselves in its implementation of the processor instruments (PI):

- the need for a significant expansion of the RAM memory PI for storing zero spectrum values;
- conducting non-productive calculations of PI with zero spectrum values;
- fixed time step in the discretization.

The purpose of this work

The purpose of this paper is to develop a method for eliminating the above mentioned shortcomings of signal interpolation in the time domain using a modified parametric discrete Fourier transform.

In the papers [1, 9–16], a generalization of the inverse DFT (IDFT) (1) in the form of a modified parametric discrete Fourier transform (MDFT-P), which is essentially the evolution of the IDFT, is proposed.

MDFT-P in the matrix form is described by the following relation [1]:

$$X_{N,\xi} = \frac{1}{N} F_{N,\xi} S_N^C, \quad 0 \le \xi < 1; \tag{7}$$

where the vector of N-dimensional linear space is the result of multiplying the second half of the vector S_N (1) by a factor:

$$C = W_N^{-(M-N)\xi},$$

where $X_{N,\xi}$ is the vector of expansion coefficients S_N^C in the system of modified parametric discrete exponential functions (MDEF-P):

$$def_{pM}\left(k,n,\xi\right) = W_N^{-k(n+\xi)} = \exp\left[+j\frac{2\pi}{N}k\left(n+\xi\right)\right],$$

$$k,n = \overline{0,\left(N-1\right)}, \ 0 \le \xi < 1,$$

which is given by the matrix $F_{N,\xi}$:

$$F_{N,\xi} = \begin{array}{c|c} & \begin{array}{ccccc} 0 & 1 & \cdots & N-1 & k \end{array} \\ \begin{array}{c} 0 \\ 1 \\ . \\ . \\ N-1 \\ n \end{array} & \begin{bmatrix} 1 & W_N^{-\xi} & \cdots & W_N^{-\xi(N-1)} \\ 1 & W_N^{-(1+\xi)} & \cdots & W_N^{-(1+\xi)(N-1)} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ 1 & W_N^{-(N-1+\xi)} & \cdots & W_N^{-(N-1+\xi)(N-1)} \end{bmatrix} \end{array}. \tag{8}$$

MAIN PROPERTIES OF MDEF-P

1. MDEF-P, like DEF-P, are not functions of equivalent variables k and n. Consequently, the MDEF-P matrix $F_{N,\xi}$ is asymmetric.

2. MDEF-P are periodic in a variable n and parametrically periodic in a variable k with a period N:

$$def_{pM}\left(k,\left(n\pm pN\right),\xi\right) = def_{pM}\left(k,n,\xi\right),$$
$$def_{pM}\left(\left(k\pm pN\right),n,\xi\right) = def_{pM}\left(k,n,\xi\right)W_N^{\pm\xi\cdot N\cdot p}.$$

3. MDEF-P system is non-multiplicative in the variable n:

$$def_{pM}\left(k,n,\xi\right) def_{pM}\left(k,m,\xi\right) \ne def_{pM}\left(k,\left(n+m\right),\xi\right), \ n,m = \overline{0,N-1}; \ n \ne m$$

and multiplicative in the variable k:

$$def_{pM}\left(k,n,\xi\right) def_{pM}\left(l,n,\xi\right)=def_{pM}\left(\left(k+l\right),n,\xi\right),\ k,l=\overline{0,N-1};\ k\neq l.$$

4. Average value of MDEF-P with respect to the variable n is equal to zero when $k \neq 0$:

$$\sum_{n=0}^{N-1} def_{pM}\left(k,n,\xi\right)=\exp\left(-j\frac{2\pi}{N}\xi k\right)\frac{1-\exp\left(-j2\pi k\right)}{1-\exp\left(-j\frac{2\pi}{N}k\right)},$$

and with respect the variable k is not zero:

$$\sum_{k=0}^{N-1} def_{pM}\left(k,n,\xi\right)=\frac{1-\exp\left(-j2\pi k\left(n+\xi\right)\right)}{1-\exp\left(-j\frac{2\pi}{N}\left(n+\xi\right)\right)}.$$

5. The MDEF-P system is orthogonal in both variables:

$$\sum_{k=0}^{N-1} W_N^{-(n+\xi)k}\left[W_N^{-(m+\xi)k}\right]^*=\frac{1-W_N^{-(n-m)N}}{1-W_N^{-(n-m)}}=\begin{cases}N, & n=m\\ 0, & n\neq m;\end{cases}$$

$$\sum_{n=0}^{N-1} W_N^{-(n+\xi)k}\left[W_N^{-(n+\xi)l}\right]^*=\frac{1-W_N^{-(l-k)N}}{1-W_N^{-(l-k)}}=\begin{cases}N, & l=k\\ 0, & l\neq k\end{cases}$$

6. The MDEF-P system is a complete system, since the number of linearly independent functions is equal to the dimension of the set of discrete signals.

Conclusion

Applying the modified parametric discrete transformation, we have the opportunity to eliminate the following essential deficiencies of the zero-padding technique, applied to the spectrum of a discrete signal $x(n)$ with N samples in the frequency domain:

- the need for a significant expansion of the RAM memory PI for storing zero spectrum values;
- Conducting non-productive calculations of PI with zero spectrum values;
- fixed time step in the discretization.

With the advent of fast hardware DSP chips and pipelined parametric FFT (FFT-P) techniques, the above time-domain interpolation algorithm may be viable for a number of applications, such as computing selectable sample rate time sequences of a test signal that has a fixed spectral envelope shape; providing interpolation, by selectable factors, of signals that were filtered in the frequency domain using the fast convolution method; or digital image resampling.

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